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Pulses of tunable size near a subcritical bifurcation

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Abstract. We show that a nonlinear gradient term can be used to tune the width of pulse-like solutions to a generalized quintic Ginzburg-Landau equation. We investigate the dynamics of these solutions and show that weakly turbulent patches can persist for long times. Analogies with turbulent spots in plane Couette flows are discussed.

PACS. 47.20.-k Hydrodynamic stability – 47.54.+r Pattern selection; pattern formation

1 Introduction

The existence of turbulent patches in otherwise laminar flows has been puzzling scientists for many years [1–6]. Little is known about these solutions, except that in open shear flows, they can either appear downstream as an amplification of upstream noise [7–9] or be triggered by a local perturbation of finite amplitude. In this latter case, experimental work on the plane Couette flow [5,6,10,11] and pipe flows [12] has revealed a critical amplitude of perturbation, A_c, for turbulent patches to grow. There appears to be a critical Reynolds number Re_c below which the flow remains stable, independently of the amplitude of the localized perturbation chosen to destabilize it. For Reynolds numbers ($Re > Re_c$) close to this threshold value, the growth of turbulent spots depends very strongly on the size of the perturbation. If Re is large, however, a finite perturbation of exceedingly low amplitude may be enough to trigger the transition [13]. From a phenomenological point of view, turbulent spots can be seen as resulting from the competition between two coexisting states [14], a turbulent and a laminar one. In this sense, they are localized solutions of a bistable system.

Model equations like the quintic subcritical Ginzburg-Landau equation

$$\frac{\partial A}{\partial t} = \mu A + (\alpha_r + i\alpha_i)\frac{\partial^2 A}{\partial x^2} + (\beta_r + i\beta_i)|A|^2 A + (\gamma + i\delta)|A|^4 A, \qquad (1)$$

have been suggested as prototypes for studying the competition between two metastable states of a same system. This equation indeed exhibits a saddle-node bifurcation which occurs for values of the control parameter below the subcritical bifurcation, thereby leading to the coexistence of three branches of solutions, two of which are linearly stable. Moreover, Thual and Fauve [15] showed that localized solutions of equation (1) connecting $A = 0$ back to $A = 0$ could be stabilized over a range of the control parameter μ. The existence of these pulses is due to non-variational effects and can be understood as resulting from a nonlinear coupling between the phase gradients and the amplitude of the control parameter A [15]. The existence of these solutions can also be inferred by considering equation (1) above as a perturbation of the Nonlinear Schrödinger equation or of the Ginzburg-Landau equation with real coefficients (see [16–19] for further detail).

One might view these pulses as being made of two kinks connecting two stable homogeneous solutions of the system. Since they exist over a finite range of μ, this contrasts with the real case ($\alpha_i = \beta_i = \delta = 0$) for which there is only one value μ_M of μ where the two homogenous solutions ($A = 0$ and $A \neq 0$) can coexist in a stable manner. This can be seen by considering the potential function associated with equation (1) with real coefficients. When $\mu = \mu_M$, the system is at its *Maxwell point*, *i.e.* the two homogeneous solutions have the same energy. When $\mu \neq \mu_M$, in one space dimension and in the absence of constraining boundary conditions, the system generically evolves towards its state of lower energy, therefore ruling out the existence of stable localized solutions. In two or higher dimensions, this trend can be counterbalanced by surface tension effects. Unless otherwise specified, our discussion will be made for one-dimensional systems.

Because they correspond to patches of non-zero $|A|$ which stably exist on a flat background, the pulses found by Thual and Fauve are natural candidates to model

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turbulent spots. Parity breaking versions of equation (1) have been considered in order to analyze noise amplification due to advection [20] and it was shown that symmetry-breaking nonlinear gradient terms could modify the propagation velocity of the pulses, as well as their shape [21]. More recently, Deissler and Brand [22] showed that the Thual-Fauve pulses [15] could become time-dependent and exhibit periodic, quasi-periodic, or chaotic behaviors. One could then think of modeling a turbulent patch by taking a Thual-Fauve pulse in a parameter regime such that the non-zero solution reached inside the pulse is phase unstable. Due to this instability, the size of the localized solution would vary in time, and a crucial question is whether or not the pulse would retain its global shape. Our numerical simulations [23] show that this is not the case and instead the pulse breaks into a few pulses, which then recombine or get apart from one another. We believe that this is due to the fact that the typical width of each pulse is strongly constrained by the Ginzburg-Landau parameters. As a consequence, the model needs to be modified if one wants to account for large turbulent patches.

In order to describe spiral turbulence [24] in Taylor-Couette flows, Hayot and Pomeau [25] introduced an integral term of the form $A\int|A|^2\ dx$. The latter can be thought of as renormalizing the control parameter μ and changing it into an effective one, whose value depends on the size of the localized solution (*i.e.* on the width of the region where $|A|\neq 0$). This term is justified in the case of annular flows but turbulent patches may nevertheless exist in non-periodic domains. The goal of this paper is to show that solutions of variable size can be stabilized by means of a local term, namely $|\partial A/\partial x|^2 A$. This term is of order 5 if one considers the Ginzburg-Landau equation as an amplitude equation and is also the nonlinear gradient term of lowest order which does not break parity symmetry. It is dominant where the amplitude or the phase of A have large gradients, and, if the real part of its coefficient has the proper sign, it can be expected to renormalize the control parameter μ in a way which may stabilize localized solutions.

In Section 2, we analyze the role of this term in the case of the real quintic Ginzburg-Landau equation. This allows us to build an intuition regarding how this new term leads to the existence of localized solutions for arbitrary values of μ. In Section 3, we give numerical results which relate the size of the localized solutions to the coefficient in front of this new term. In Section 4, we consider localized solutions for which the non-zero state is phase unstable and show that one can observe, at least over long periods of time, turbulent patches on top of a zero (or laminar) background. Finally, we discuss possible analogies of this system with turbulent spots observed in a plane Couette experiment [10,26].

2 Localized solutions to the quintic Ginzburg-Landau equation with real coefficients

We are interested in the existence of localized solutions to the complex Ginzburg-Landau equation with real coefficients, which reads

$$\frac{\partial A}{\partial t}=\mu A+\alpha_r\frac{\partial^2 A}{\partial x^2}+\beta_r|A|^2A+\gamma|A|^4A+\zeta\left|\frac{\partial A}{\partial x}\right|^2A. \tag{2}$$

We assume that $\alpha_r>0$, $\beta_r>0$ and $\gamma<0$, which insures the existence of a subcritical bifurcation at $\mu=0$. For values of μ between $\mu_{SN}=\beta_r^2/(4\gamma)$ and zero, this equation possess three stationary solutions of the form $A=\pm R$, where $R=0$ or $R=R^{\pm}$ is one of the non-negative roots of $R(\gamma R^4+\beta_r R^2+\mu)=0$. One can then expect to find real localized solutions connecting $R=0$ to $R\neq 0$. The existence of such solutions can be inferred from the phase diagram of the spatial dynamical system obtained from (2) by setting the temporal derivative of A equal to zero. We then look for kinks connecting $A=0$ to a state of non-zero amplitude. A pulse would correspond to a homoclinic solution starting and ending at $A=0$, but the above equation does not posses such solutions. However, because interactions between kinks are expected to be exponential, one can construct localized solutions by juxtaposing two kinks, one connecting $A=0$ to $A\neq 0$, and another one connecting $A\neq 0$ back to $A=0$ [27].

The spatial dynamical system describing stationary solutions of (2) has five critical points, given by $R=0$ and $R=\pm R^{\pm}$. Two of them ($\pm R^-$) are centers and three of them ($R=0$ and $R=\pm R^+$) are saddles. The phase diagrams for typical values of μ are sketched in Figure 1. Only the first quadrant is shown. The others quadrants are obtained by symmetry. When $\zeta=0$, there is only one value of μ, corresponding to the Maxwell point $\mu_M=3\beta_r^2/(16\gamma)$, for which there exists a heteroclinic connection between $R=0$ and $R=\pm R^+$ (see Fig. 1b). One can clearly see in Figure 1c how the heteroclinic orbit starting at $R=-R^+$ and ending at $R=R^+$ collides with the stable and unstable manifolds of $R=0$ when μ crosses μ_M from above.

When $\zeta\neq 0$, the term in $\zeta|\partial A/\partial x|^2A$ can be seen as renormalizing μ by a quantity equal to $\zeta|\partial A/\partial x|^2$, which has the same sign as ζ. This suggests that an appropriate choice of ζ may lead to the existence of a heteroclinic connection between $R=0$ and $R=\pm R^+$, for every μ. Note that the critical points $\pm R^{\pm}$ as well as their linear stability are not affected by ζ. When $\mu<\mu_M$, the value of ζ giving the existence of a heteroclinic connection is positive, since the actual μ needs to be "increased" towards μ_M. Similarly, if $\mu>\mu_M$, ζ must be chosen negative in order to make a heteroclinic connection possible. Below, we use phase-space considerations to prove the existence and uniqueness of this heteroclinic connection when ζ is varied.

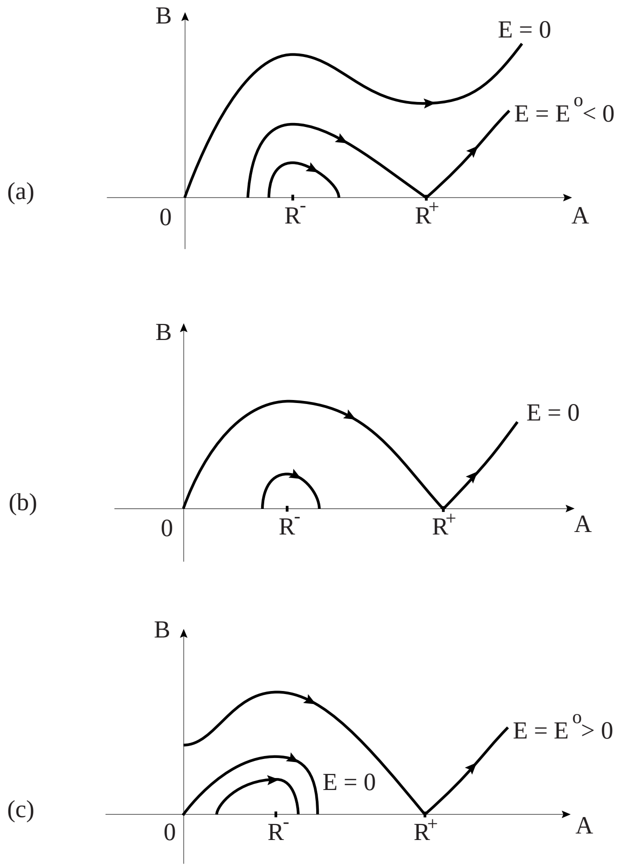

Fig. 1. Phase diagrams for the spatial dynamical system corresponding to the time-independent Ginzburg-Landau equation with real coefficients. Only the first quadrant is shown. (a) $\mu < \mu_M$; (b) $\mu = \mu_M$; (c) $\mu > \mu_M$.

2.1 Existence and uniqueness of a heteroclinic connection for every μ

We assume that μ is negative, but larger than $\mu_{SN} = \beta_r^2/4\gamma$, in order to grant the existence of five fixed points. When A is real and time-independent, equation (2) can be written as

$$\begin{aligned} \frac{dA}{dx} &= B \\ \frac{dB}{dx} &= -\frac{1}{\alpha_r}\left[\mu A + \beta_r A^3 + \gamma A^5 + \zeta B^2 A\right] \\ &= -\frac{1}{\alpha_r}\left[f(A) + \zeta B^2 A\right], \end{aligned} \tag{3}$$

where $f(A) = \mu A + \beta_r A^3 + \gamma A^5$. We define the "energy"

$$E = \frac{B^2}{2} + \int_0^A \frac{1}{\alpha_r}\, f(v)\; dv.$$

It can be easily checked that

$$\frac{dE}{dx} = -\frac{\zeta}{\alpha_r} B^3 A.$$

In other words, the energy is conserved on the solution curves when $\zeta = 0$. Another way of looking at Figure 1 is to view it as a contour diagram of the energy E in the first quadrant for three different values of μ ($\mu < \mu_M$, $\mu = \mu_M$ and $\mu > \mu_M$), when $\zeta = 0$. In each plot, two level curves play an important role: the level curve $E = 0$, which corresponds to the unstable manifold of the origin, and the level curve $E = E^o$, which corresponds to the stable manifold of the fixed point $(R^+, 0)$. The equation of this latter curve is

$$B^2 = \frac{-\gamma}{3\alpha_r}\left((R^+)^2 - A^2\right)^2 (A^2 - z_0^2),$$

where

$$z_0^2 = (R^-)^2 + \frac{\sqrt{\beta_r^2 - 4\gamma\mu}}{2\gamma},$$

and z_0^2 is positive if and only if $\mu < \mu_M$. In other words, the curve $E = E^o$ crosses the positive A-axis in two different points when $\mu < \mu_M$, and in only one point when $\mu > \mu_M$, as can be seen in Figure 1.

When $\mu \neq \mu_M$ and $\zeta = 0$, the unstable manifold of $(0, 0)$ is, in the first quadrant, above the stable manifold of $(R^+, 0)$ if $\mu < \mu_M$ (Fig. 1a), and below the stable manifold of $(R^+, 0)$ if $\mu > \mu_M$ (Fig. 1c). Our proof for the existence and unicity of a heteroclinic connection consists in showing that one can find a value of ζ, positive in the former case and negative in the latter, for which the situation is reversed, *i.e.* such that the unstable manifold of $(0, 0)$ is below the stable manifold of $(R^+, 0)$ if $\mu < \mu_M$, and above the stable manifold of $(R^+, 0)$ if $\mu > \mu_M$. By continuity of the solution curves with the parameters, this means that, for each μ, there is a value ζ_c of ζ for which the unstable manifold of $(0, 0)$ coincides with the stable manifold of $(R^+, 0)$, *i.e.* for which a heteroclinic connection exists. The proof is given in Appendix. We now compute the first order term of the expansion of ζ_c as a function of the difference $\mu - \mu_M$.

2.2 Perturbation of the heteroclinic orbit for μ near μ_M and small ζ

When $\mu = \mu_M$, we know that the heteroclinic connection occurs for $\zeta = \zeta_c(\mu_M) = 0$. Assuming that $\zeta_c(\mu)$ is smooth, one can compute the first order term of the Taylor expansion of ζ as a function of $\mu - \mu_M$ by writing a solvability condition for the existence of a heteroclinic orbit when $\mu \simeq \mu_M$.

When $\mu = \mu_M$, the heteroclinic solution between $A = 0$ and $A = R^+$ can be found by quadrature and is given by

$$A(x) = A_M(x) = \left[\frac{\beta_r}{\frac{-4\gamma}{3} + C\beta_r \exp\left(-\sqrt{-3/\alpha_r\gamma}\,\beta_r x/2\right)}\right]^{1/2},$$

where C is an arbitrary constant, corresponding to different choices of the origin of the space variable x. We now

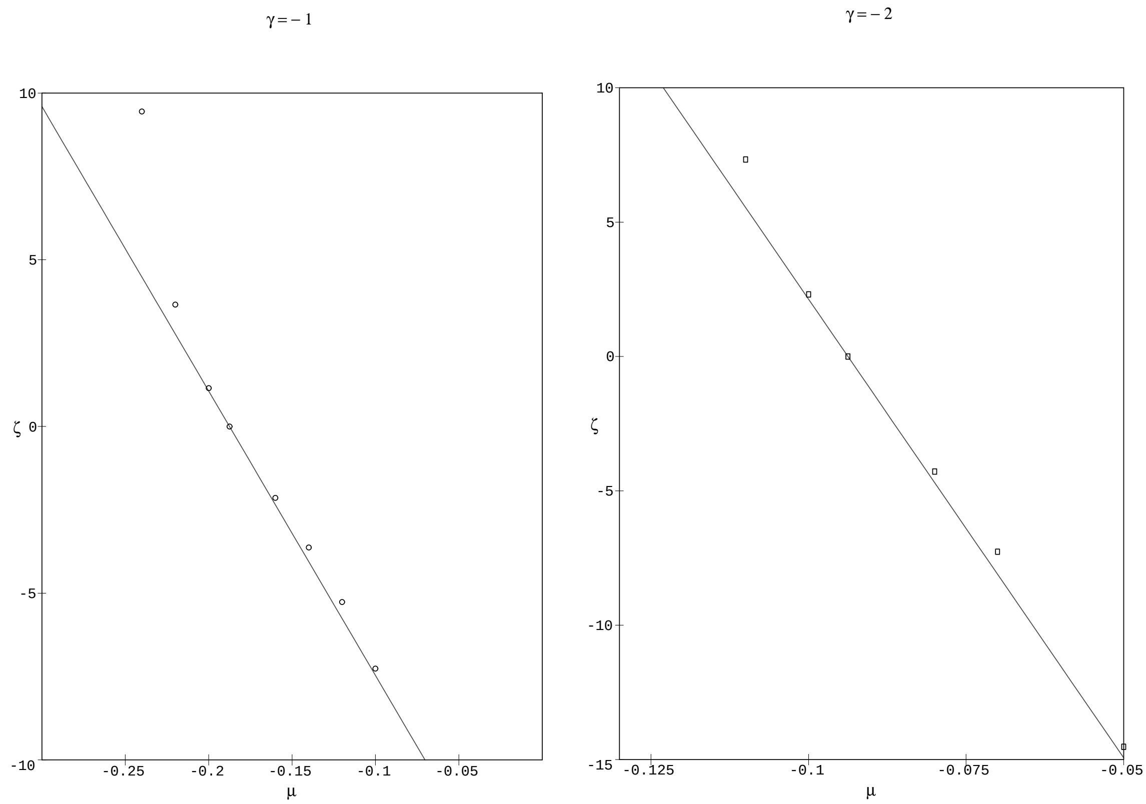

Fig. 2. Plot of the first order approximation of ζ_c as a function of μ (solid line), together with numerically measured values of ζ_c (circles and squares), for $\alpha_r = \beta_r = 1$ and $\gamma = -1$ (left) and $\gamma = -2$ (right).

assume $\mu = \mu_M + \epsilon$, where ϵ is small, and $\zeta = \epsilon\zeta_1 + O(\epsilon^2)$. We look for a solution to equation (2) in the form

$$A = A_M + \epsilon A_1 + O(\epsilon^2),$$

and obtain, at order ϵ, the following linear equation for A_1:

$$\mathcal{L}A_1 = -A_M \left(\zeta_1 \left(\frac{dA_M}{dx} \right)^2 + 1 \right) \tag{4}$$

where

$$\mathcal{L}(u) = \left(\mu_M + \alpha_r \frac{d^2}{dx^2} + 3\beta_r A_M^2 + 5\gamma A_4^4 \right) u.$$

The first order correction A_1 to the heteroclinic connection is bounded but not zero at $+\infty$, since

$$\lim_{x\to+\infty} A_1 = \frac{1}{\epsilon} \left[R^+(\mu) - R^+(\mu_M) \right] = \frac{1}{2R^+(\mu_M)(\beta_r - 4\gamma\mu_M)^{1/2}}.$$

Since $\mathcal{L}$ is self-adjoint for the usual scalar product, by multiplying both sides of equation (4) by dA_M/dx and integrating between $-\infty$ and $+\infty$, one gets, assuming that $dA_1/dx \to 0$ when x goes to $\pm\infty$,

$$\int_{-\infty}^{+\infty} \left(\zeta_1 \left(\frac{dA_M}{dx} \right)^3 A_M + \frac{dA_M}{dx} A_M \right) dx = 0,$$

which gives

$$\zeta_1 = 1024 \frac{\gamma^3}{9\beta_r^4} (R^+)^2,$$

i.e., with $(R^+)^2 = -3\beta_r/(4\gamma)$,

$$\zeta_c = -\frac{256}{3} \frac{\gamma^2}{\beta_r^3} (\mu - \mu_M) + O\left((\mu - \mu_M)^2 \right). \tag{5}$$

Note that ζ_c does not depend on α_r since this parameter can be scaled out of the equation without changing the value of ζ. Figure 2 shows the first order approximation given by equation (5) together with measured values of ζ_c for $\alpha_r = \beta_r = 1$ and two values of γ: $\gamma = -1$ (for which $\mu_M = -0.1875$ and $\mu_{SN} = -0.25$) and $\gamma = -2$ (for which $\mu_M = -0.09375$ and $\mu_{SN} = -0.125$). The value of ζ_c was measured numerically using the dynamical systems software DSTOOL. The agreement is extremely good, even for relatively large values of $\mu - \mu_M$.

The above analysis allows us to understand the role of the nonlinear gradient term: we can tune its coefficient to create heteroclinic connections between $A = 0$ and $A = \pm R^+$ for arbitrary values of μ. As a consequence, long-lived localized solutions corresponding to the juxtaposition of exponentially-interacting symmetric kinks may be observed. When complex coefficients are taken into account, these solutions are not only stable, but their size varies greatly with the coefficient of the nonlinear gradient

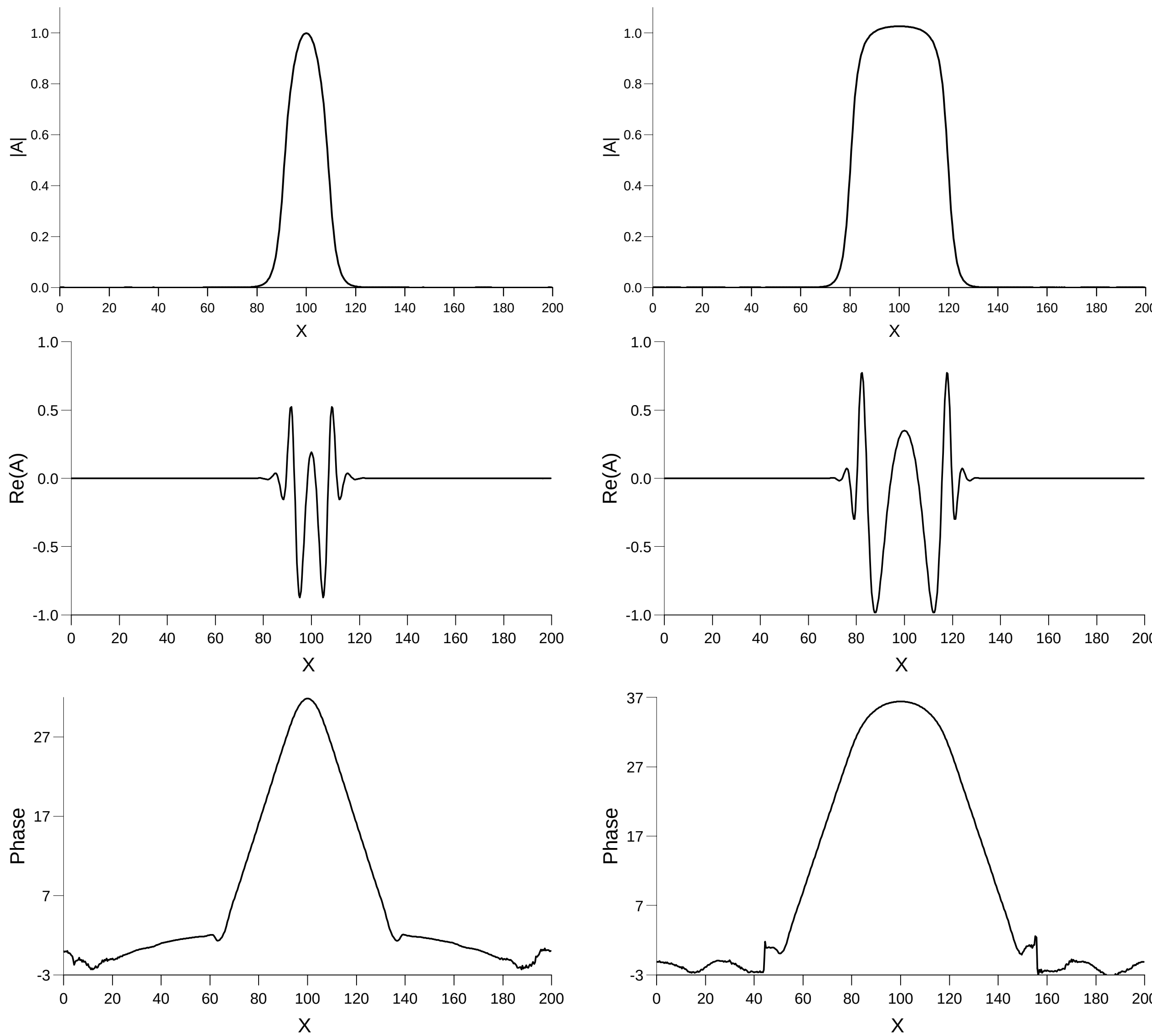

Fig. 3. Localized solutions to the quintic complex Ginzburg-Landau equation (6) with (left) $\zeta = \kappa = 0$ and (right) $\zeta = -0.133$ and $\kappa = 0.88$. Other parameters are $\alpha_r = 1.2$, $\alpha_i = -1.1$, $\beta_r = 3.0$, $\beta_i = 1.0$, $\gamma = -2.75$ and $\delta = 1.0$.

term. This parameter may then be used to "tune" the width of these localized objects, as discussed below.

3 Localized solutions of tunable size

Here we give numerical results showing localized solutions to the quintic complex Ginzburg-Landau equation (with complex coefficients)

$$\frac{\partial A}{\partial t} = (\mu + i\nu)A + (\alpha_r + i\alpha_i)\frac{\partial^2 A}{\partial x^2} + (\beta_r + i\beta_i)|A|^2 A + (\gamma + i\delta)|A|^4 A + (\zeta + i\kappa)\left|\frac{\partial A}{\partial x}\right|^2 A. \quad (6)$$

We have used a spectral code with periodic boundary conditions. Our simulations were made in boxes of various sizes in order to check that the width of the localized solutions was independent of the size of the system. The amplitude, real part and phase of a Thual-Fauve pulse [15] are shown in Figure 3, as functions of space. Left plots correspond to (6) with $\zeta = \kappa = 0$, and right ones to (6) with $\zeta = -0.133$ and $\kappa = 0.88$. The other parameters are the same as those of [22], that is $\alpha_r = 1.2$, $\alpha_i = -1.1$, $\beta_r = 3.0$, $\beta_i = 1.0$, $\gamma = -2.75$ and $\delta = 1.0$. It is obvious from this picture that the size of the localized solution is larger when ζ and κ are not zero. The new solution has the same characteristic features as the Thual-Fauve pulses, except that it is wider.

We now discuss how the size of these pulses is affected by the values of ζ and κ. For this purpose, we use a scaled Ginzburg-Landau equation, *i.e.* (6) with $\alpha_r = 1$

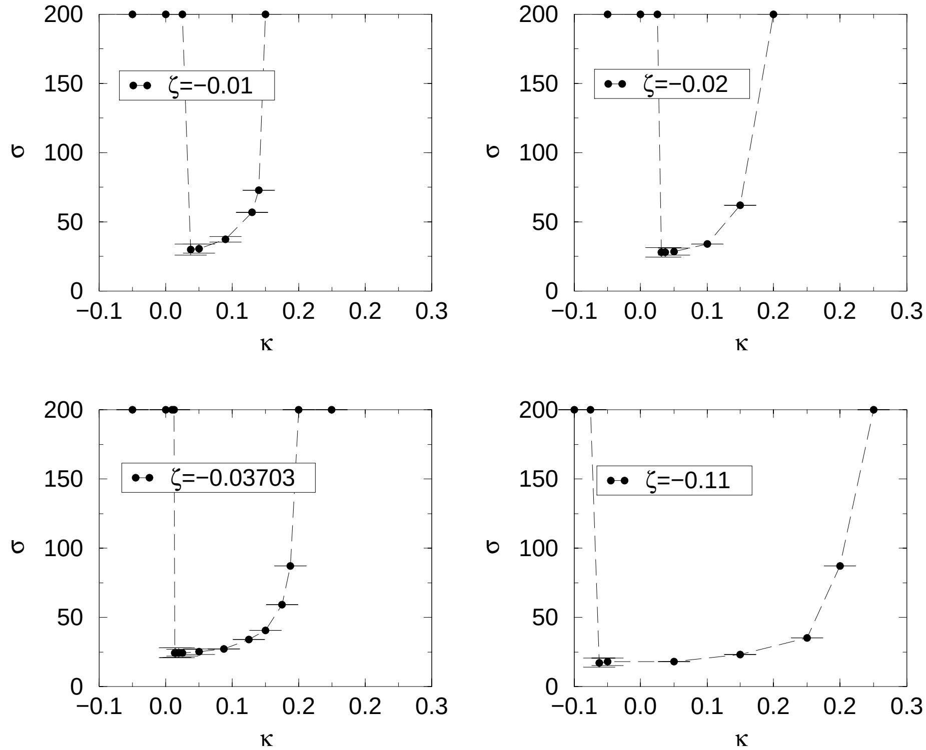

Fig. 4. Typical evolution of the size of the localized solution as a function of κ for fixed values of ζ.

and $\beta_r = 1$. Figure 4 gives the width of the localized solution as a fuction of κ for fixed values of ζ. As predicted, the size of the pulse is completely determined by the value of κ (all other parameters being fixed) and can be tuned by changing this parameter. Stable solutions exist in a finite range of κ's, denoted $[\kappa_-,\ \kappa_+]$. If κ is outside this interval, any localized solution evolves towards a homogeneous state characterized by $|A|$ = constant. When κ is between κ_- and κ_+, localized solutions can be viewed as "patches of non-zero $|A|$" within a laminar region where $A = 0$. Moreover, the order parameter A behaves like a plane wave in the region where $A \neq 0$. These solutions of (6) are of the form

$$A = R\exp[i(qx + \omega(q)t)]$$

where

$$\begin{cases} \omega(q) = \nu - \alpha_i q^2 + (\beta_i + \kappa q^2)R^2 + \delta R^4 \\ 0 = \mu - \alpha_r q^2 + (\beta_r + \zeta q^2)R^2 + \gamma R^4 \end{cases} . \tag{7}$$

When κ is increased towards κ_+, the plane wave solution reached within the pulse takes over $A = 0$ and spreads over the whole box. When κ is decreased towards κ_-, this solution becomes phase unstable [28,29] and the pulse eventually loses stability. The system then converges towards another plane wave of smaller wave number, which is phase (or Eckhaus [30]) stable.

The linear stability of a plane wave solution (7) can be investigated by computing its phase diffusion (or Cross-Newell [31]) equation. It reads

$$\frac{\partial\theta}{\partial t} = \omega(q) + E\frac{\partial^2\theta}{\partial x^2} + h.o.t.$$

where $\omega(q)$ is given in (7),

$$\begin{aligned} E = {} & \frac{2\alpha_r q^2}{R^2}\frac{\alpha_r - \zeta R^2}{2\gamma R^2 + \beta_r + \zeta q^2} + \alpha_r \\ & + \frac{\beta_i + 2\delta R^2 + \kappa q^2}{\beta_r + 2\gamma R^2 + \zeta q^2}\left[\frac{2q^2}{R^2}\frac{\alpha_r - \zeta R^2}{2\gamma R^2 + \beta_r + \zeta q^2}\right. \\ & \left.\times\left[\kappa R^2 + (\alpha_r - \zeta R^2)\frac{\beta_i + 2\delta R^2 + \kappa q^2}{\beta_r + 2\gamma R^2 + \zeta q^2}\right] + \alpha_i\right], \end{aligned} \tag{8}$$

θ is the phase of the order parameter A, and $h.o.t$ stands for higher order terms. Plane waves are unstable when the "diffusion coefficient" E is negative. Our numerics shows that the wave number selected by the pulse becomes Eckhaus unstable when κ is decreased towards κ_-. Moreover, there exists a range of values of κ for which the pulse, although phase unstable, remains localized for very long periods of time. Traveling holes [32], which are deep and narrow amplitude depressions, are then seen to propagate within the pulse, leading to a complex dynamical behavior which we discuss below.

4 Dynamical behaviors

The phase instability of the solution reached within the pulse acts as a source of noise, which makes the size of the pulse time-dependent. In the absence of the nonlinear gradient term $|\partial A/\partial x|^2 A$, Deissler and Brand [22] had shown that the envelope of the pulse could be made

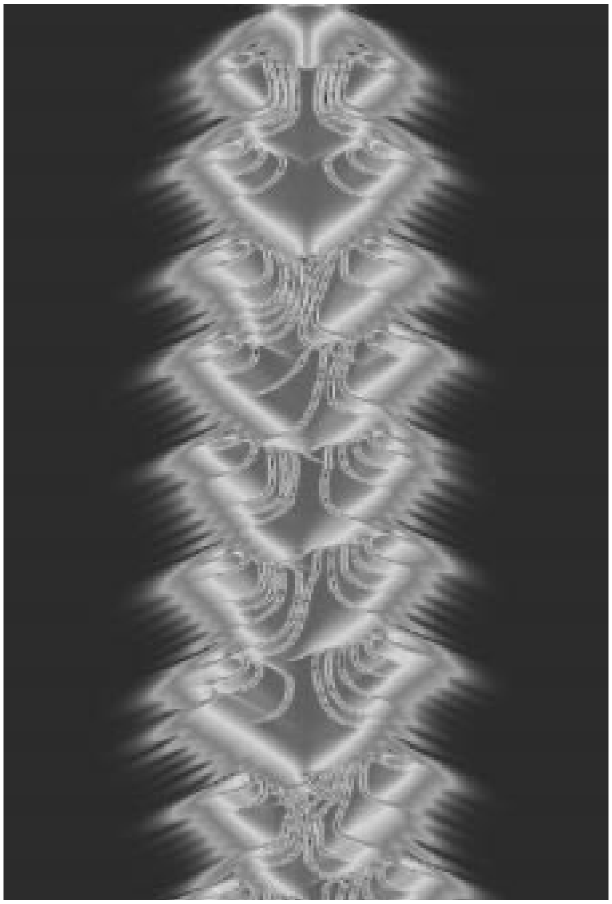

Fig. 5. Spatio-temporal diagram showing the amplitude ($|A|$) of a localized solution to (6) with a time-dependent envelope. Traveling holes are clearly seen to propagate within the pulse. The parameters used to generate this picture are $\alpha_r = 1$, $\alpha_i = -11/3$, $\beta_r = 1$, $\beta_i = 1/3$, $\gamma = -2.75/9$, $\delta = 1/9$, $\zeta = -14/9$ and $\kappa = -0.8$. The box length (horizontal axis) is 400 units long and the vertical axis corresponds to 400 units of time. Time goes downward.

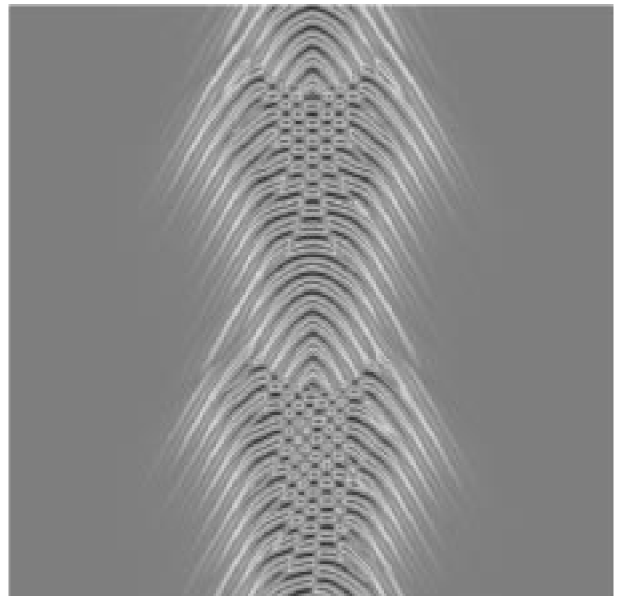

Fig. 6. Spatio-temporal diagram showing the real part of a localized solution with a time-dependent envelope. The two time scales corresponding to the period of the plane wave and to the modulation of the envelope of the pulse are clearly visible. The parameters used to generate this picture are $\alpha_r = 1$, $\alpha_i = -11/3$, $\beta_r = 1$, $\beta_i = 1/3$, $\gamma = -2.75/9$, $\delta = 1/9$, $\zeta = -14/9$ and $\kappa = -0.8$. The box length (horizontal axis) is 400 units long and the vertical axis corresponds to 100 units of time. Time goes downward.

time-periodic, quasi-periodic or even chaotic by an appropriate choice of α_r. Here, because the nonlinear gradient term can stabilize pulses of rather large size, complicated spatio-temporal dynamics can take place inside the pulse. As shown in Figure 5, traveling holes [32] are created near the center of the pulse, propagate towards the edges, and at the same time the width of the pulse shows a strong time-dependence. Such solutions persist over extremely long periods of time, which are at least a few hundred times as long as the characteristic time scale given by the frequency ω of the plane wave. Figure 6 shows the real part of a "turbulent pulse", where the two time scales can be seen. The dark and light stripes which, at the center of the pulse, are almost parallel to the horizontal axis correspond to oscillations of the solution at the frequency ω. The time-scale at which the width of the pulse evolves is about 20 times as large. The whole picture describes the solution over 100 units of time, whereas Figure 5 covers 400 units of time. The typical length of each of our runs was around 2000 units of time.

The above pulses can be seen as localized patches of turbulence, moving in a homogeneous (*i.e.* laminar) background. It is very likely that the two-dimensional version of the Ginzburg-Landau equation (6) will possess localized solutions similar to our one-dimensional pulses, and analogies can be drawn between these solutions and turbulent spots seen in shear flows, such as the plane Couette flow analyzed in [5,6,10,26]. This flow has no mean advection, is linearly stable, and may sustain turbulent spots. The latter possess a characteristic size which for instance varies by as much as 40% when the Reynolds number, which is the experimental control parameter, is changed from 320 to 380 [33]. These spots may interact if two of them happen to get close enough to one-another. In this case, they coalesce and create a large spot, whose size eventually relaxes to that of a single spot [34]. As a test for our model, we chose an initial condition with two gaussian shapes. In the absence of the other pulse, each of these profiles would evolve toward a single pulse solution, similar to those described above. As shown in Figure 7, the two localized perturbations first coalesce into a large pulse (Fig. 7, left plot), which then relaxes towards a single localized solution (Fig. 7, right plot), whose size is the same as if we had started with a single gaussian perturbation.

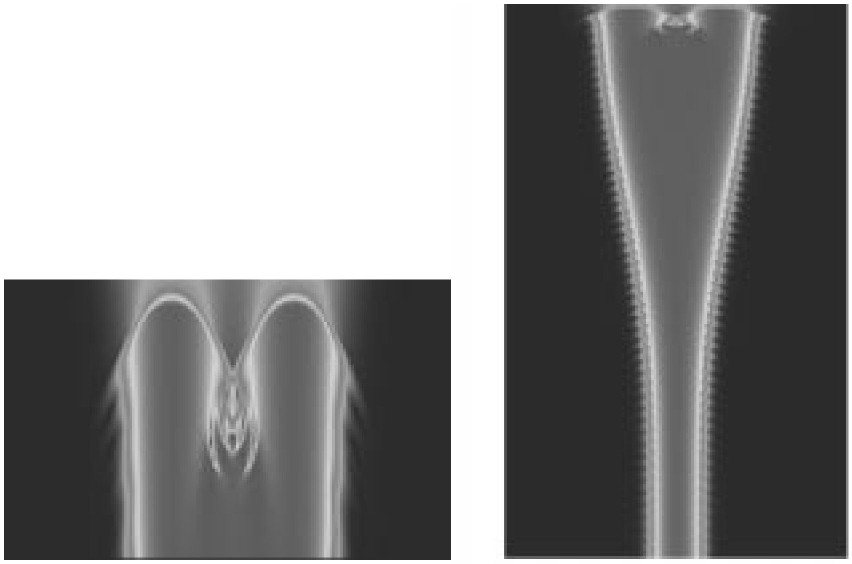

Fig. 7. Spatio-temporal diagrams showing the evolution of two localized perturbations. Left: in the initial stage, the two localized solutions expand, collide and unite into a single pulse (box length = 200 units of space; the vertical axis corresponds to 25 units of time; time goes downward). Right: the pulse relaxes to its characteristic size (box length = 200 units of space; the vertical axis corresponds to 400 units of time; time goes downward).

5 Conclusions

We have shown that a nonlinear gradient term can stabilize localized solutions of tunable size, including weakly turbulent ones. This term does not break the parity invariance of the Ginzburg-Landau equation and is of same order as the saturating quintic term $|A|^4A$. Since its presence has significant consequences on the width of the localized solutions, it should not be omitted in the description of subcritical instabilities. Such a term would indeed generically appear in Ginzburg-Landau equations describing inverted bifurcations in spatially extended systems, or associated with model equations like the complex quintic Swift-Hohenberg (SH) equation. In the case of a real SH equation, the parameter κ would be zero and localized solutions would be described by equation (2). The fact that ζ can always be chosen to ensure the existence of a heteroclinic connection between $A = 0$ and $A \neq 0$ may explain the localized solutions observed by Sakagushi and Brand [35] in the quintic SH equation. Finally, the generalized Ginzburg-Landau equation (6) provides a good model for the description of turbulent patches in laminar domains, and we believe that its extension to two space dimensions will give a qualitatively good description of experimentally observed turbulent spots.

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Appendix: Proof of the existence and uniqueness of a heteroclinic connection for $\zeta = \zeta_c(\mu)$

The proof relies on two preliminary results, one ensuring that each solution curve in the first quadrant can be thought of as the graph of a function $B = h_\zeta(A)$, the other one showing that solution curves going through the same point but for different values of ζ cross only once. Each of these properties is discussed below.

1. *Proposition 1: Consider the solution curves of System (3) in the first quadrant (i.e. A, $B \geq 0$). Each curve can be seen as the graph of a function $B = h_{\zeta(A)}$.*
   Indeed, if a solution curve had more than one intersection point with a line $A = C =$ constant in the first quadrant, there would, by continuity, exist a point where $dA/dx = B = 0$ and $B \neq 0$, which is impossible. As a consequence, we will, without loss of generality, restrict our analysis to the first quadrant and consider B as a function of A on any solution curve.
2. *Proposition 2: Two solution curves which correspond, in the first quadrant, to different values of ζ and have the same initial condition cross only once, at the initial point.*
   Consider two distinct solution curves $B = h_{\zeta_1}(A)$ and $B = h_{\zeta_2}$, $\zeta_1 \neq \zeta_2$, which both go through the point with coordinates (A_0, B_0). Without loss of generality, we can assume that h_{ζ_1} is above h_{ζ_2} for every A near A_0, *i.e.* there exists $A_1 > A_0$ such that $h_{\zeta_1}(A) > h_{\zeta_2}(A)$ for every A in $(A_0, A_1]$. Then, using

$$\frac{dE}{dA} = \frac{dE}{dx}\frac{dx}{dA} = -\frac{\zeta}{\alpha_r}B^2A,$$

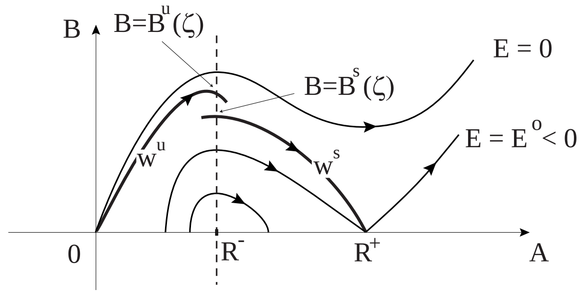

Fig. 8. Sketch of the phase portrait of system (3) when $\mu < \mu_M$.

and the definition of E, the energy difference between the two curves can be written in two ways:

$$\begin{aligned} &E\left(A, h_{\zeta_1}(A)\right) - E\left(A, h_{\zeta_2}(A)\right) \\ &= \frac{1}{2} h_{\zeta_1}(A)^2 - \frac{1}{2} h_{\zeta_2}(A)^2 \\ &= \int_{A_0}^{A} -\frac{\zeta_1}{\alpha_r} h_{\zeta_1}(A)^2 \; A \; dA - \int_{A_0}^{A} -\frac{\zeta_2}{\alpha_r} h_{\zeta_2}(A)^2 \; A \; dA. \end{aligned}$$

Thus, for $A_0 \leq A \leq A_1$,

$$\frac{1}{\alpha_r} \int_{A_0}^{A} \left[\zeta_2 h_{\zeta_2}(A)^2 - \zeta_1 h_{\zeta_1}(A)^2\right] \; dA > 0.$$

Since the integrand has a constant sign between A_0 and A_1, this implies

$$\zeta_2 h_{\zeta_2}(A)^2 > \zeta_1 h_{\zeta_1}(A)^2 \qquad \text{for } A_0 < A < A_1,$$

i.e.

$$\zeta_2 > \zeta_1 \frac{h_{\zeta_1}(A)^2}{h_{\zeta_2}(A)^2} \qquad \text{for } A_0 < A < A_1.$$

Since h_{ζ_1} is above h_{ζ_2}, we have $\zeta_2 > \zeta_1$. As a consequence, the two curves cannot cross again because if they were, there would be a point $\tilde{A}_0$ and a point $\tilde{A}_1$ such that $h_{\zeta_1}(\tilde{A}_0) = h_{\zeta_2}(\tilde{A}_1)$ and $h_{\zeta_2}(A) > h_{\zeta_1}(A)$ for $\tilde{A}_0 < A < \tilde{A}_1$. By repeating the above argument, one would then have $\zeta_1 > \zeta_2$, which contradicts the previous result.
Therefore, two solution curves which correspond, in the first quadrant, to different values of ζ and have the same initial condition cross only once, at the initial point.

We now consider an arbitrary value of μ, $\mu_{SN} < \mu < 0$, and show that there is an unique value of ζ for which a heteroclinic connection exists. Let $w^u(A,\zeta)$ denote the unstable manifold of the origin in the first quadrant, and $w^s(A,\zeta)$ the stable manifold of $(R^+, 0)$, also in the first quadrant.

1. If $\mu < \mu_M$, $w^u(A,0)$ is above $w^s(A,0)$ when $\zeta = 0$ (see Fig. 8). Let us consider positive values of ζ. Since $dE/dx < 0$ on solution curves, $w^u(A,\zeta)$ stays *below* $w^u(A,0)$. Moreover, $w^u(A,\zeta)$ cannot cross the A-axis at a value $A \leq R^-$ since one must have

$$\frac{dB}{dx} = -\frac{\gamma}{\alpha_r}\left(A^2 - (R^+)^2\right)\left(A^2 - (R^-)^2\right) A < 0$$

at the crossing point, which does not occur if $0 < A < R^-$. Thus $w^u(A,\zeta)$ crosses the line $A = R^-$ at a point with coordinates $(R^-, B^u(\zeta))$, where $B^u(\zeta) > 0$. Since for any $\zeta_1, \zeta_2 > 0$, $w^u(A,\zeta_1)$ and $w^u(A,\zeta_2)$ cross at the origin, Proposition 2 above shows that $B^u(\zeta)$ is a monotonic decreasing function of ζ.
Now consider $w^s(A,\zeta)$. When $\zeta = 0$, it crosses the line $A = R^-$ at a point with coordinates $(R^-, B^s(0))$, with $B^s(0) < B^u(0)$. When $\zeta \neq 0$, because for any $\zeta_1, \zeta_2 > 0$, $w^s(A,\zeta_1)$ and $w^s(A,\zeta_2)$ cross at $(R^+, 0)$, $B^s(\zeta)$ is a monotonic increasing function of ζ. We want to show that there is a value of ζ for which $B^s(\zeta) > B^u(\zeta)$.
The upper branch of the level curve $E = E^o$ is the stable manifold $w^s(A,0)$, and, as explained before, $w^s(A,\zeta)$ remains above this curve when $\zeta > 0$. On $w^s(A,\zeta)$,

$$\begin{aligned} \frac{dB}{dx} = &-\frac{\gamma}{\alpha_r} A\left(A^2 - (R^+)^2\right)\left(A^2 - (R^-)^2\right) \\ &-\frac{\zeta}{\alpha_r} B^2 A, \text{with} \quad B = w^s(A,\zeta). \end{aligned}$$

Since

$$\begin{aligned} &-\frac{\gamma}{\alpha_r} A\left(A^2 - (R^+)^2\right)\left(A^2 - (R^-)^2\right) \leq 0 \\ &\text{for } R^- \leq A \leq R^+, \end{aligned}$$

and since $w^s(A,0) < w^s(A,\zeta)$ and $\zeta > 0$, we have

$$\begin{aligned} &-\frac{\gamma}{\alpha_r} A\left(A^2 - (R^+)^2\right)\left(A^2 - (R^-)^2\right) - \frac{\zeta}{\alpha_r}[w^s(A,\zeta)]^2 A \\ &< -\frac{\zeta A}{\alpha_r} w^s(A,0) w^s(A,\zeta) \text{ for } R^- \leq A \leq R^+. \end{aligned}$$

Therefore, for every $\zeta > 0$,

$$\frac{dw^s(A,\zeta)}{dA} = \frac{1}{w^s(A,\zeta)} \frac{dw^s(A,\zeta)}{dx} < -\frac{\zeta}{\alpha_r} w^s(A,0) A,$$

for $R^- \leq A \leq R^+$. Let $X = R^+ - A$. With this new variable, the above equation reads

$$\begin{aligned} \frac{d\tilde{w}^s(X,\zeta)}{dX} &> \frac{\zeta}{\alpha_r} X(2R^+ - X)\sqrt{-\frac{\gamma}{3\alpha_r}\left[(R^+ - X)^2 - z_0^2\right]} \\ &= \zeta g(X) \geq 0, \end{aligned}$$

for $z_0 \leq R^+ - X \leq R^+$, and where we have defined $\tilde{w}^s(X,\zeta) = w^s(A,\zeta)$. To write this last inequality, we have used the expression of $w^s(A,0)$ as the upper branch of the level curve $E = E^o$, whose equation is given in Section 2.1. Since $R^- > z_0 > 0$, we get

$$\begin{aligned} \tilde{w}^s(R^+ - R^-, \zeta) &= w^s(R^-, \zeta) \\ &= B^s(\zeta) \geq \zeta \int_0^{R^+ - R^-} g(X) \; dX = \mathcal{C}\zeta, \end{aligned}$$

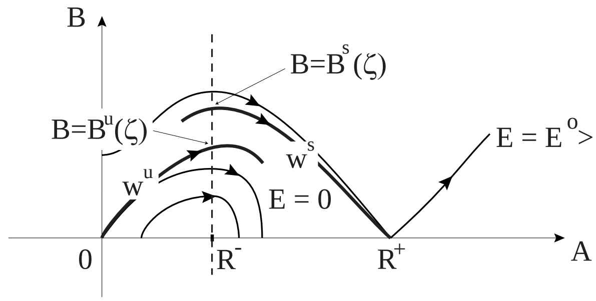

Fig. 9. Sketch of the phase portrait of system (3) when $\mu > \mu_M$.

where $\mathcal{C}$ is a strictly positive constant. By choosing ζ large enough, one can then insure that $B^s(\zeta) > B^u(0) > B^u(\zeta)$, *i.e.* that

$$B^s(\zeta) > B^u(\zeta).$$

By continuity of the solution with respect to changes in the parameters, there must be a value ζ_c for which $B^u(\zeta) = B^s(\zeta)$, *i.e.* for which there is a heteroclinic orbit. Moreover, this value is unique because B^u and B^s are monotonic functions of ζ which vary in opposite directions.

2. If $\mu > \mu_M$, $w^u(A,0)$ is below $w^s(A,0)$ when $\zeta = 0$. Let us consider negative values of ζ. Since $dE/dx > 0$ on solution curves, $w^s(A,\zeta)$ stays *below* $w^s(A,0)$, *i.e.* $B^s(\zeta) < B^s(0)$ (see Fig. 9). The points $B^s(\zeta)$ and $B^u(\zeta)$ are defined as before. From the level curves of E (see Fig. 1), we know that $B^s(0) > B^u(0)$.
We want to show that for values of $|\zeta|$ large enough, the situation is reversed, *i.e.* $B^u(\zeta) > B^s(\zeta)$. For $0 \le A \le R^-$,

$$\begin{aligned}\frac{dB}{dx} &= -\frac{\gamma}{\alpha_r} A\left(A^2 - (R^+)^2\right)\left(A^2 - (R^-)^2\right) - \frac{\zeta}{\alpha_r} B^2 A \\ &\ge -\frac{\zeta}{\alpha_r} B^2 A,\end{aligned}$$

and for $w^u(A,\zeta)$

$$\frac{dw^u(A,\zeta)}{dA} \ge -\frac{\zeta}{\alpha_r} w^u(A,\zeta) A \ge -\frac{\zeta}{\alpha_r} w^u(A,0) A,$$

since $w^u(A,\zeta)$ is above $w^u(A,0)$. Therefore, by integrating from $A = 0$ to $A = R^-$,

$$B^u(\zeta) = w^u(R^-,\zeta) \ge -\frac{\zeta}{\alpha_r}\int_0^{R^-} w^u(A,0)\ A\ dA = -\zeta\mathcal{C}^*,$$

where $\mathcal{C}^*$ is a strictly positive constant. By choosing $|\zeta| = -\zeta$ large enough, one gets

$$B^u(\zeta) > B^s(0) > B^s(\zeta).$$

Again, by continuity of the solution curves with respect to the parameters, there is a value ζ_c for which $B^u(\zeta_c) = B^s(\zeta_c)$, *i.e.* for which a heteroclinic connection exists. Since B^u and B^s are respectively increasing and decreasing monotonically when $|\zeta|$ is increased, the value ζ_c is unique.

Therefore, for every μ, there is a unique value ζ_c for which a heteroclinic connection exists. Moreover, if $\mu < \mu_M$, ζ_c is strictly positive; if $\mu > \mu_M$, ζ_c is strictly negative.

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